MTH 303
Real analysis **Homework 6**

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that
$$|f(x) - f(y)| \leq (x - y)^2$$
for all x and y in $\mathbb{R}$. Show that f is constant.

2. Let $f : (a, b) \to \mathbb{R}$ be differentiable at $x \in (a, b)$. Show that
$$\lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h} = f'(x).$$
Show by an example that the above limit may exist even if $f'(x)$ does not.

3. Let $f : (a, b) \to \mathbb{R}$ be twice differentiable at $x \in (a, b)$. Show that
$$\lim_{h \to 0} \frac{f(x + h) + f(x - h) - 2f(x)}{h^2} = f''(x).$$
Show by an example that the above limit may exist even if $f''(x)$ does not.

4. If $f(x) = |x|^3$ for all $x \in \mathbb{R}$, compute $f'(x)$ and $f''(x)$ for each $x \in \mathbb{R}$. Does $f^{(3)}(0)$ exist?

5. Show that $f(x) = x|x|$ is differentiable on $\mathbb{R}$. Compute $f'(x)$. Is f' continuous? Does f'' exist?

6. Show that each of the following functions are infinitely differentiable on $\mathbb{R}$.
   (a)
$$f(x) = \begin{cases} 0, & x \leq 0 \\ e^{-\frac{1}{x}}, & x > 0 \end{cases}$$
   (b)
$$f(x) = \begin{cases} 0, & x \leq 0 \\ e^{-\frac{1}{x^2}}, & x > 0 \end{cases}$$

7. Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Define $g(x) = f(x + 1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

8. Let $f : (a, b) \to \mathbb{R}$ a differentiable function such that $f'(x) \neq 0$ for all $x \in (a, b)$. Show that f is monotone on (a, b).

9. Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be the inverse function of f. Prove that g is differentiable, and that
$$g'(f(x)) = \frac{1}{f'(x)} \quad \text{for all } x \in (a, b).$$

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $|f'(x)| \leq M$ for some $M > 0$ for all $x \in \mathbb{R}$.
    (a) Show that f is uniformly continuous on $\mathbb{R}$.
    (b) If $\epsilon > 0$ is sufficiently small, then show that the function $g(x) = x + \epsilon f(x)$ is one-one.

11. Let $f : [a, b] \to \mathbb{R}$ be such that $f'(x) \neq 0$ for all $x \in (a, b)$. Show that $f(a) \neq f(b)$.

12. (Cauchy's form of MVT) Let $f, g : [a, b] \to \mathbb{R}$ be differentiable. Assume that $g'(x) \neq 0$ for each $x \in (a, b)$, the prove that there exists $c \in (a, b)$ such that
$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

13. Use MVT to prove the following inequalities.
    (a) For $0 < a < b$, show that $b^{\frac{1}{n}} - a^{\frac{1}{n}} < (b - a)^{\frac{1}{n}}$.
    (b) $|\sin x - \sin y| \leq |x - y|$.
    (c) $1 + x < e^x$ for all $x > 0$.

14. If
$$c_0 + \frac{c_1}{2} + \frac{c_2}{3} + \cdots + \frac{c_n}{n+1} = 0,$$
where $c_0, c_1, \ldots, c_n$ are real constants, prove that the equation
$$c_0 + c_1 x + \cdots + c_n x^n = 0$$
has at least one real root in $(0, 1)$.

15. Show that the function $f(x) = x^3 - 3x^2 + 17$ is not one-one on the interval $[-1, 1]$.

16. Show that the equation $x^3 - 3x + 17$ has at most one root in the interval $[-1, 1]$.

17. If f is differentiable on $[a, b]$, then show that f cannot have any simple discontinuity on $[a, b]$.

18. Let f be a continuous real-valued function on $\mathbb{R}$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exist?

19. Use Taylor's formula to prove that
    (a) $\log 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$ (Hint: Consider $f(x) = \log(1 + x)$).
    (b) $e = \sum_{k=0}^{\infty} \frac{1}{k!}$
    (c) $1 - \frac{x^2}{2} < \cos x$ for all $x \in \mathbb{R}$ (Hint: Consider $f(x) = \cos x$).

20. (L'Hospital's Rule) Suppose f and g are differentiable in (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$, where $-\infty \leq a < b \leq +\infty$. Suppose

$$\frac{f'(x)}{g'(x)} \to A \text{ as } x \to a.$$

If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ or if $g(x) \to +\infty$ as $x \to a$, then show that

$$\frac{f(x)}{g(x)} \to A \text{ as } x \to a.$$

21. Use L'Hospital's Rule to show that
    (a) $\lim\limits_{x\to\infty} \frac{\log x}{x} = 0$
    (b) $\lim\limits_{x\to\infty} \frac{x^n}{e^x} = 0$ where $n \in \mathbb{N}$.

22. Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number M such that $|f'(x)| \leq M|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

23. If $f : (a, b) \to \mathbb{R}$ is convex and $c \in (a, b)$ is a local mimimum, then show that c is a global minimum for f.